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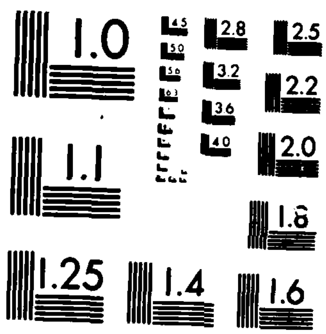

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THE QUASI-STATIC PROPAGATION OF A PLANE STRAIN CRACK IN A POWER-LAW INHOMOGENEOUS LINEARLY VISOELASTIC BODY

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The Quasi-Static Propagation of a Plane Strain Crack in a Power-Law Inhomogeneous Linearly Viscoelastic Body

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Abstract

An analysis is presented of the steady-state propagation of a semi-infinite mode I crack for an infinite inhomogeneous, linearly viscoelastic body. The shear modulus is assumed to have a power-law dependence on depth from the plane of the crack. Moreover, both a general and a power-law behavior in time for the shear modulus are considered. A simple closed form expression for the normal component of stress in front of the propagating crack is derived which exhibits explicitly the form of the stress singularity and its material dependency. The crack profile is examined and its dependence on the spatial and time behavior of the shear modulus is determined.

1. Introduction.

In this paper an analysis is presented of the quasi-static problem of a semi-infinite mode I crack propagating through an infinite, inhomogeneous, isotropic linearly viscoelastic body. Viscoelastic models involving both a general and a power-law behavior in time for the shear modulus are considered.

One of the few investigations of a boundary value problem involving a nonhomogeneous viscoelastic body was carried out by Walton (1984) in which a quasi-static punch problem was addressed assuming a shear modulus exhibiting a power-law time and depth dependence. This investigation was carried out by first appealing to the correspondence principle and then reducing the boundary value problem to a generalized Abel integral equation. An analysis of the dynamic steady state propagation of a semi-infinite anti-plane shear crack in a general infinite homogeneous viscoelastic body was presented by Walton (1982). Under very weak assumptions on the shear modulus the problem was reformulated as a Riemann-Hilbert problem and ultimately simple closed form expressions were constructed for the stress intensity factor and the entire stress field in front of the advancing crack. The approach adopted in this study employs methods from both papers. The correspondence principle is utilized to derive an integral relation between the normal components of the stress and displacement fields and this relationship is subsequently reduced to a Riemann-Hilbert problem.

Before proceeding to the formulation of the boundary value problem under consideration, a few comments on the viscoelastic models assumed in this work are in order. The homogeneous power-law viscoelastic model may be characterized by a shear modulus, μ, of the form

$$\mu(t) = \mu_c(t/t_c)^{-\alpha}$$

where t denotes time, t_c is a characteristic relaxation time, and μ_c is a characteristic modulus. Clearly, this simple power-law behavior in time introduces certain unphysical features into the model. However, as discussed by Walton (1984) and Walton et.al. (1978) and references cited therein, the pure power-law model can yield useful engineering predictions for many real materials.

In addition to the power-law model, a general form for the time dependence of the shear modulus is also considered. More specifically, the only restrictions placed on the time dependence of the shear modulus are that it be a positive, continuous, decreasing, and convex function of time. Though the solution of the Riemann-Hilbert problem requires a separate analysis depending on the particular form of the time dependence assumed for the shear modulus, the normal component of the stress field in front of the crack is identical in both cases. Indeed, the entire stress field is independent of any time dependence assumed for the modulus.

In this paper the viscoelastic body in which the crack propagates is assumed to be inhomogeneous in that the shear modulus has a power-law depth dependence. In particular the shear modulus $\mu(t,y)$ has the form

$$\mu(t,y) = \mu_c g(t)(y/y_c)^{\gamma}$$

where y denotes distance as measured from the plane of the crack, y_c is a characteristic depth, $0 \leq \gamma \leq 1$, and g(t) satisfies the conditions discussed above. Again, it may at first appear that a model with a simple power-law depth dependence is physically unreasonable and

therefore only of mathematical interest. However, as pointed out by Walton (1984) there are materials whose mechanical behavior obeys such a power-law model over very broad ranges of time and depth. Though the materials and examples cited concerning a power-law depth dependence occur more naturally within the realm of punch problems, there are scenarios for which the power-law model applies to the physical situations that arise in the case of a crack in a viscoelastic material. For instance, boundary value problems involving thermorheologically simple viscoelastic materials with a spatially dependent but time independent temperature field are of the type amenable to analysis under isothermal but nonhomogeneous spatial assumptions. Within this context the power-law model may provide useful insight when the mechanical deformation associated with the propagation of the crack produces a significant thermal gradient which in turn results in a depth-dependent softening of the material. Other physical situations in which the power-law model is appropriate occur when a crack propagates in a polymer between laminated materials where moisture has lead to a softening of the polymer material and in certain instances of hydraulic fracturing.

In the next section the transformation of the governing boundary value problem to a Riemann-Hilbert problem is presented and the solution of the Riemann-Hilbert problem is derived. In Section 3 an analysis of the stress field is carried out and it is shown that the usual square-root singularity associated with the stress near the crack tip is altered by the inhomogeneity of the material. A very simple closed form expression for the normal component of stress is derived which exhibits explicitly the nature of the singularity and its material

dependency and indicates that the usual notion of the stress intensity factor must be modified. The section closes with an asymptotic analysis of the crack profile near its tip. It is observed that the displacements of the crack surfaces depend on both the spatial and time behavior of the shear modulus.

2. Formulation of the Problem.

The boundary value problem to be studied is that of a mode I semi-infinite crack propagating with constant speed in a nonhomogeneous isotropic viscoelastic media in a state of plane strain. Our ultimate goal is to describe the normal component of stress in front of the advancing crack and the nature of the crack profile. If we assume the crack propagates in a plane about which the spatial properties of the body are symmetric and along the x_1-axis with speed v, driven by loads $f(x_1-vt)$ which follow it, then the specific boundary value problem to be solved is

$$\sigma_{ij,j} = 0 \quad -\infty < x_1 < \infty, x_2 > 0$$

$$\sigma_{ij} = 2\mu^* d\varepsilon_{ij} + \delta_{ij}\left(\frac{2\nu}{1-2\nu}\right)\mu^* d\varepsilon_{kk}$$

$$\sigma_{22}(x_1,0,t) = f(x_1-vt) \quad x_1 < vt$$

$$u_2(x_1,0,t) = 0 \quad x_1 > vt$$

$$\sigma_{12}(x_1,0,t) = 0 \quad -\infty < x_1 < \infty. \tag{1}$$

Here $\sigma_{ij}, \varepsilon_{ij}$, and u_i are the viscoelastic stress, strain, and displacement fields, ν is Poisson's ratio (assumed to be constant), μ is the shear modulus, δ_{ij} is the Kronecker delta, and $\mu^* d\varepsilon$ denotes the

Riemann-Stieltjes convolution

$$\mu * d\varepsilon = \int_{-\infty}^{t} \mu(t-\tau)\, d\varepsilon(\tau).$$

The construction of the solution of the preceding equations is facilitated by adoption of the Galilean variable $x = x_1 - vt$ and the change of variables $y_1 = y, \sigma_{11} = \sigma_{xx}, \sigma_{12} = \sigma_{xy}, u_1 = u_x$, etc. For the shear modulus, two cases are considered:

case (1) $\mu_1(t,y) = \mu_c g(t)(y/y_c)^{\gamma}$

case (2) $\mu_2(t,y) = \mu_c (t/t_c)^{-\alpha}(y/y_c)^{\gamma}$.

In both cases $0 \leq \gamma < 1$ and in case (1) $g(t)$ is assumed to be positive, continuously differentiable, nonincreasing, convex, and such that $\mu(\infty) = \lim_{t\to\infty} \mu(t) > 0$.

A key step towards determining σ_{yy} and u_y is the construction of the so-called transfer function, i.e., the function $T(p)$ for which

$$\hat{\sigma}_{yy}(p,0) = T(p)\hat{u}_y(p,0)$$

where $\hat{f}(p,y)$ denotes the Fourier transform

$$\hat{f}(p,y) = \int_{-\infty}^{\infty} f(x,y) e^{ipx} dx.$$

The derivation of the transfer function is carried out as in Walton (1984). A Fourier transform in x is applied to the boundary value problem and then we appeal to the viscoelastic correspondence principle. Use can then be made of the known integral boundary relation between elastic stresses and displacements for a power-law depth-dependent shear modulus. Specifically, it is shown in Gladwell (1980)

that

$$u_y^e(x,0) = \int_{-\infty}^{\infty} [g_{12}\mathrm{sgn}(x-t)\sigma_{xy}^e(t,0) - g_{22}\sigma_{yy}^e(t,0)]\,|x-t|^{-\gamma}dt \tag{2}$$

where u_y^e, σ_{xy}^e, and σ_{yy}^e are the elastic displacements and stresses and $\mathrm{sgn}(\cdot)$ denotes the signum function. The constants g_{12} and g_{22} are given by

$$g_{12} = (1-\nu)I\cos(\pi q/2)/(\pi\mu_c^e\gamma)$$

$$g_{22} = (1-\nu)Iq\sin(\pi q/2)/(\pi\mu_c^e\gamma(1+\gamma))$$

$$q = [(1+\gamma)(1-\gamma\nu/(1-\nu))]^{1/2}$$

$$I = 2^\gamma(\gamma+2)B(\frac{\gamma+q+3}{2}, \frac{\gamma-q+3}{2}).$$

The elastic shear modulus adopted by Gladwell has the form

$$\mu^e(y) = \mu_c^e y^\gamma.$$

The viscoelastic transfer function may now be derived by Fourier transform of (2) and substitution of the transformed viscoelastic shear modulus for μ_c^e. (Recall that x is the Galilean variable x_1-vt. Thus $\mu(t,y)$ is a function of x and y and $\hat{\mu}$ denotes its Fourier transform with respect to x.) If use is made of (1), it can be shown in this way that the transfer functions for cases (1) and (2) are as follows:

$$\text{case (1)}\quad \hat{\sigma}_{yy}(p,0) = -K_o[(ipv)\hat{g}(-vp)]\,|p|^{1-\gamma}\hat{u}_y(p,0) \tag{3}$$

$$\text{case (2)}\quad \hat{\sigma}_{yy}(p,0) = -K_o[(vt_c)^\alpha\Gamma(1-\alpha)(ip)^\alpha]\,|p|^{1-\gamma}\hat{u}_y(p,0) \tag{4}$$

where

$$K_o = \frac{\mu_c\Gamma(\gamma+2)\cos(\gamma\pi/2)}{q(1-\nu)I\sin(q\pi/2)y_c^\gamma}.$$

After introduction of the functions

$$H_1(p) = (ipv)\hat{g}(-vp) = g(0) + \int_0^\infty e^{-ivpt} dg(t) \tag{5}$$

and

$$H_2(p) = (vt_c)^\alpha \Gamma(1-\alpha)(ip)^\alpha = (vt_c)^\alpha \Gamma(1-\alpha)|p|^\alpha e^{i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)},$$

(3) and (4) may be written as

$$\hat{\sigma}^+_{yy} + \hat{\sigma}^-_{yy} = -K_o H_i(p)|p|^{1-\gamma}\hat{u}^-_y,\ i=1,2. \tag{6}$$

Here σ^+_{yy} and σ^-_{yy} denote the restriction to the positive and negative real axes respectively of σ_{yy}. Similarly for u^+_y and u^-_y. Line (6) may be viewed as the Riemann-Hilbert problem

$$F^+(p) = -K_o H_i(p)|p|^{1-\gamma}F^-(p) + h(p),\ i=1,2 \tag{7}$$

where

$$F^+(p) = \hat{\sigma}^+_{yy}(p,0) = \int_0^\infty \sigma_{yy}(x,0)e^{ipx}dx,$$

$$F^-(p) = \hat{u}^-_y(p,0) = \int_{-\infty}^0 u_y(x,0)e^{ipx}dx,$$

and

$$h(p) = -\hat{\sigma}^-_{yy}(p,0) = -\int_{-\infty}^0 f(x)e^{ipx}dx.$$

It is assumed a priori (and it may be verified a posteriori) that σ^+_{yy} and u^-_y are such that F^+ and F^- define functions analytic in the upper and lower half-planes respectively. Moreover, the limits

$$F^+(p) = \lim_{q\to 0^+} F^+(p+iq),\ F^-(p) = \lim_{q\to 0^-} F^-(p+iq)$$

both exist and satisfy (7).

Attention is now directed to the solution of the Riemann-Hilbert problems (7). (The reader who is unfamiliar with the theory of Riemann-Hilbert boundary value problems may wish to consult Gakov (1966).) To solve

$$F^+(p) = -K_o H_1(p)|p|^{1-\gamma} F^-(p) + h(p) \tag{8}$$

it is useful to consider first the homogeneous problem of finding functions $\Phi^{\pm}(z)$ analytic for $\mathrm{Im}\, z \gtrless 0$, respectively, which satisfy the homogeneous boundary relation

$$\Phi^+(p) = -K_o H_1(p)|p|^{1-\gamma}\Phi^-(p). \tag{9}$$

It suffices to solve separately

$$\Phi_1^+(p) = H_1(p)\Phi_1^-(p) \tag{10}$$

and

$$\Phi_2^+(p) = -K_o|p|^{1-\gamma}\Phi_2^-(p), \tag{11}$$

for then $\Phi^{\pm}(z) = \Phi_1^{\pm}(z)\Phi_2^{\pm}(z)$ is a solution to (9). In order to solve (10) it is necessary to determine the mapping properties of $H_1(p)$. With the stated assumptions for $g(t)$, it is shown (Walton, 1982) that

i) $H_1(0) = g(\infty) \le \mathrm{Re}\, H_1(-vp) \le g(0) = H_1(\infty)$

ii) $\mathrm{Im}\, H_1(-vp) = -\mathrm{Im}\, H_1(vp)$

iii) $\arg H_1(-vp)\begin{cases} \ge 0 & p > 0 \\ \le 0 & p < 0 \end{cases}$.

From these properties it follows that the image of H_1 in the complex plane of the real p-axis is as depicted in Fig. 1. In particular $H_1(p)$ is Hölder continuous for all p on the extended real line so that the canonical solution to (10) is given by

$$\Phi_1^{\pm}(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln H_1(\tau)}{\tau-z}\,d\tau\right].$$

Though the usual methods of solving a Riemann-Hilbert problem may not be applied to (11), since the coefficient function vanishes at the origin and becomes infinite at ∞, a solution is easily found by inspection. Indeed, if

$$\Phi_2^{+}(z) = c^{+}z^{(\frac{1-\gamma}{2})} \quad \text{and} \quad \Phi_2^{-}(z) = c^{-}z^{-(\frac{1-\gamma}{2})},$$

then

$$\Phi_2^{+}(p) = \frac{c^{+}}{c^{-}}\,|p|^{1-\gamma}\Phi_2^{-}(p). \tag{12}$$

Hence (12) is a solution to (11) provided that $c^{+}/c^{-} = -K_o$, and

$$\Phi^{\pm}(z) = c^{\pm}z^{\pm(\frac{1-\gamma}{2})}\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln H_1(\tau)}{\tau-z}\,d\tau\right]$$

is therefore a solution to (9).

Before proceeding to construct a solution to the nonhomogeneous Riemann-Hilbert problem (8), it should be observed that $\Phi^{+}(z)$ can be greatly simplified. From (5) and the assumptions on g(t) it follows that $H_1(p)$ has a natural extension $H_1(p+iq)$ for $q < 0$. Moreover

i) $H_1(\pm\infty + iq) = H_1(\infty)$,

ii) $H_1(iq_1) < H_1(iq_2)$, $q_2 < q_1 < 0$,

and

iii) $\lim\limits_{q\to-\infty} H_1(iq) = H_1(\infty)$.

Fig. 1 depicts the mapping properties of $H_1(z)$ for $\operatorname{Im} z < 0$. Clearly, for $\operatorname{Im} z > 0$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln H_1(\tau)}{\tau - z}\, d\tau = \frac{1}{2}\ln H_1(\infty) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln H_1(\tau) - \ln H_1(\infty)}{\tau - z}\, d\tau. \tag{13}$$

From the stated properties of $H_1(p-iq)$, $q>0$, it follows that $|\ln H_1(\tau) - \ln H_1(\infty)| \to 0$ as $|\tau| \to \infty$ and hence that the integral in (13) vanishes by an application of Cauchy's theorem. Thus

$$\Phi^+(z) = c^+ z^{\left(\frac{1-\gamma}{2}\right)} \sqrt{g(0)}$$

and

$$\Phi^+(p) = c^+ \sqrt{g(0)}\; k(p)\, |p|^{\frac{1-\gamma}{2}} \tag{14}$$

where

$$k(p) = \begin{cases} 1 & p > 0 \\ e^{i\frac{\pi}{2}(1-\gamma)} & p < 0. \end{cases} \tag{15}$$

The solution to the Riemann-Hilbert problem (8) in case (1) is now determined by the expression

$$F^\pm(z) = \Phi^\pm(z)\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\tau)}{\Phi^+(\tau)}\,\frac{d\tau}{\tau - z}\,.$$

Application of the Plemelj formulas together with (14) and (15) yields

$$F^+(p) = \frac{1}{2}h(p) + \frac{k(p)|p|^{\frac{1-\gamma}{2}}}{2\pi i}\int_{-\infty}^{\infty}\frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}}\frac{d\tau}{\tau-p}\,. \tag{16}$$

Attention will now be focused upon case (2), which corresponds to the power-law time dependent model. To solve

$$F^+(p) = -K_o H_2(p)|p|^{1-\gamma}F^-(p)+h(p), \tag{17}$$

a solution to

$$\Psi^+(p) = -K_o H_2(p)|p|^{1-\gamma}\Psi^-(p) \tag{18}$$

is sought once again in the form $\Psi^{\pm} = \Psi_1^{\pm}\Psi_2^{\pm}$ where

$$\Psi_1^+(p) = -K_o(vt_c)^{\alpha}\Gamma(1-\alpha)|p|^{1+\alpha-\gamma}\Psi^-(p) \tag{19}$$

and

$$\Psi_2^+(p) = e^{i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)}\,\Psi_2^-(p)\,. \tag{20}$$

Problem (19) may be solved as before, producing the solution

$$\Psi_1^{\pm}(z) = c^{\pm}z^{\pm(\frac{1+\alpha-\gamma}{2})}$$

where

$$c^+/c^- = -K_o(vt_c)^{\alpha}\Gamma(1-\alpha)\,. \tag{21}$$

A solution of (20) may be constructed by defining

$$\alpha(z) = \ln(z) \quad 0 < \arg z < 2\pi$$

and

$$\beta(z) = \ell n(z) \quad -\pi < \arg z < \pi,$$

$z = p+iq$, and setting

$$\delta^{\pm}(z) = (\alpha^{\pm}(z) + \beta^{\pm}(z))/2\pi i \tag{22}$$

where $\alpha^{\pm}$, $\beta^{\pm}$ denote the restrictions of α and β to the upper and lower half-planes respectively. Then $\delta^{\pm}(z)$ is analytic for $\operatorname{Im} z \gtrless 0$ respectively and it can be verified that

$$\delta^{+}(p)-\delta^{-}(p) = -\operatorname{sgn}(p).$$

Hence

$$\Psi_{2}^{\pm}(z) = \exp[-i\alpha \tfrac{\pi}{2} \delta^{\pm}(z)]$$

provides a solution of (20). Thus a solution of (18) is given by

$$\Psi^{\pm}(z) = c^{\pm} z^{\pm(\frac{1+\alpha-\gamma}{2})} \exp[-i\alpha \tfrac{\pi}{2} \delta^{\pm}(z)]$$

where c^{+} and c^{-} satisfy (21) and δ^{+} and δ^{-} are defined in (22). The calculation of $\hat{\sigma}_{yy}^{+}(p,0)$ requires that $\Psi^{+}(p)$ be known, and it is easy to see that

$$\begin{aligned}
\Psi^{+}(p) &= c^{+}|p|^{\frac{1+\alpha-\gamma}{2}} e^{-i\alpha \frac{\pi}{2} \delta^{+}(p)} \begin{Bmatrix} 1 & p>0 \\ e^{i\frac{\pi}{2}(1+\alpha-\gamma)} & p<0 \end{Bmatrix} \\
&= c^{+}|p|^{\frac{1+\alpha-\gamma}{2}} |p|^{-\alpha/2} \begin{Bmatrix} 1 & p>0 \\ e^{-i\alpha\frac{\pi}{2}} & p<0 \end{Bmatrix} \begin{Bmatrix} 1 & p>0 \\ e^{i\frac{\pi}{2}(1+\alpha-\gamma)} & p<0 \end{Bmatrix} \\
&= c^{+}k(p)|p|^{\frac{1-\gamma}{2}}
\end{aligned} \tag{22}$$

where $k(p)$ is defined in (15).

Thus the solution of the Riemann-Hilbert problem (17) is

$$F^{\pm}(z) = \Psi^{\pm}(z)\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\tau)}{\Psi^{+}(\tau)}\,\frac{d\tau}{\tau - z}\,.$$

Again, making use of the Plemelj formulas and (23) it follows that for case (2), $F^{+}(p)$ is given by (16). Since $F^{+}(p) = \hat{\sigma}^{+}_{yy}(p,0)$, it is clear that for both cases (1) and (2),

$$\hat{\sigma}^{+}_{yy}(p,0) = \frac{1}{2}h(p) + \frac{k(p)\,|p|^{\frac{1-\gamma}{2}}}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)\,|\tau|^{\frac{1-\gamma}{2}}}\,\frac{d\tau}{\tau - p}. \tag{24}$$

Denote by $f^{\vee}$ the inverse Fourier transform, that is

$$f^{\vee}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp} f(p)\,dp.$$

Since $h(p) = -\hat{\sigma}^{-}_{yy}(p,0)$, Fourier inversion of (24) yields for $x > 0$,

$$\sigma^{+}_{yy}(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}\,\frac{k(p)\,|p|^{\frac{1-\gamma}{2}}}{2\pi i}\int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)\,|\tau|^{\frac{1-\gamma}{2}}}\,\frac{d\tau}{\tau - p}\quad dp. \tag{25}$$

Again, it should be observed that (25) gives the normal component of stress in front of the crack for both cases (1) and (2). In particular it is clear that the stress depends only on the spatial properties exhibited by the viscoelastic shear modulus. In the next section, (25) is written in a very simple form which explicitly displays the nature of the singularity of $\sigma^{+}_{yy}(x)$ near the crack tip. Futhermore, the crack profile is examined.

3. Stress and Displacement Analysis.

In order to analyze $\sigma^{+}_{yy}(x)$ for all $x > 0$ the following well known identity relating the Fourier and Hilbert transforms is recalled:

$$H[f] = [-i\ \mathrm{sgn}\,(\tau)\,f^{\vee}(\tau)]^{\vee} \tag{26}$$

where

$$H[f] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-\tau}\,dt.$$

Define functions α and β by

$$\hat{\alpha}(\tau) = \frac{k(\tau)\,\mathrm{sgn}(\tau)}{|\tau|^{\frac{1+\gamma}{2}}} \tag{27}$$

$$\hat{\beta}(\tau) = \frac{1}{k(\tau)\,|\tau|^{\frac{1-\gamma}{2}}}, \tag{28}$$

with $k(\tau)$ given by (15). From (26) - (28) and the definitions of $h(p)$, the expression for $\sigma^{+}_{yy}(x)$ given by (25) may be written as

$$\sigma^{+}_{yy}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}\ \frac{k(p)\,|p|^{\frac{1-\gamma}{2}}}{2i}\ H[-\hat{\sigma}_{yy}\hat{\beta}](p)\quad dp$$

$$= \frac{1}{2\pi}\frac{d}{dx}\int_{-\infty}^{\infty} e^{-ixp}\ \frac{k(p)\,|p|^{\frac{1-\gamma}{2}}}{2p}[i\ \mathrm{sgn}\,(\tau)\,(\sigma^{-}_{yy}*\beta)\,(\tau)\,]^{\wedge}(p)\quad dp$$

$$= \frac{i}{2}\,\frac{d}{dx}[\hat{\alpha}(p)\,[\mathrm{sgn}(\tau)\,(\sigma^{-}_{yy}*\beta)\,(\tau)\,]^{\wedge}(p)\,]^{\vee}(x)$$

$$= \frac{i}{2}\,\frac{d}{dx}[\alpha(\tau)*[\mathrm{sgn}(\tau)\,(\sigma^{-}_{yy}*\beta)\,(\tau)\,]\,](x). \tag{29}$$

(Here * denotes the convolution $(f*g)(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau$.) From (27) $\alpha(x)$ is calculated as

$$\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k(\tau)\,\mathrm{sgn}(\tau)}{|\tau|^{\frac{1+\gamma}{2}}} e^{-i\tau x} d\tau$$

$$= \frac{\Gamma(\frac{1-\gamma}{2})}{2\pi|x|^{\frac{1-\gamma}{2}}} \begin{cases} e^{-i\frac{\pi}{4}(1-\gamma)}[1+e^{-i\pi\gamma}] & x>0 \\ 0 & x<0 \end{cases} . \tag{30}$$

Similarly for $\beta(x)$,

$$\beta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}} d\tau$$

$$= \frac{\Gamma(\frac{1+\gamma}{2})}{2\pi|x|^{\frac{1+\gamma}{2}}} \begin{cases} e^{-i\frac{\pi}{4}(1+\gamma)}[1+e^{i\pi\gamma}] & x>0 \\ 0 & x<0 \end{cases} . \tag{31}$$

By utilizing (30) and (31), interchanging the order of integration in (29), and simplifying the coefficient, the stress for $x>0$ may be written as

$$\sigma^{+}_{yy}(x) = \frac{\cos(\frac{\gamma\pi}{2})}{2\pi}\int_{-\infty}^{0}\sigma^{-}_{yy}(s)\left\{\frac{d}{dx}\int_{s}^{x}\frac{\mathrm{sgn}(\tau)}{(x-\tau)^{\frac{1-\gamma}{2}}(\tau-s)^{\frac{1+\gamma}{2}}}d\tau\right\} ds.$$

After setting $\tau = s+t(x-s)$ and performing the differentiation, the normal stress ahead of the crack is easily seen to be given by

$$\sigma^{+}_{yy}(x) = \frac{\cos(\frac{\gamma\pi}{2})}{\pi x^{(1-\gamma)/2}} \int_{-\infty}^{0} \sigma^{-}_{yy}(s)\,|s|^{\frac{1-\gamma}{2}} \frac{ds}{s-x}, \quad x > 0. \tag{32}$$

In view of the simple form of (32) certain observations are apparent. By setting $\gamma=0$ the expression for the known stress in front of a uniformly moving crack in a homogeneous elastic body is recovered (Willis, 1967). Near the crack tip, as $x \to 0^{+}$,

$$\sigma^{+}_{yy}(x) \sim \frac{K(\gamma)}{x^{(1-\gamma)/2}}$$

where

$$K(\gamma) = -\frac{\cos(\frac{\pi\gamma}{2})}{\pi} \int_{-\infty}^{0} \frac{\sigma^{-}_{yy}(s)}{|s|^{\frac{1+\gamma}{2}}}\,ds. \tag{33}$$

The effect of the parameter γ is to reduce the order of the usual square root singularity near the crack tip. This seems physically plausible in view of the implications of a vanishing shear modulus in the crack plane. The coefficient $K(\gamma)$ in (33) may still be called a stress intensity factor in the sense that it is the coefficient of the leading singular term of the asymptotic expansion of the stress at the crack tip.

To illustrate the behavior of $K(\gamma)$, consider the case in which the crack is driven by constant tractions on a finite interval traveling with the crack tip, i.e., $\sigma^{-}_{yy}(x,0) = -P[H(-x)-H(-a-x)]$ where a and P are constants and H is the Heaviside function. Then

$$K(\gamma) = \frac{2P}{(1-\gamma)\pi} \cos(\frac{\gamma\pi}{2}) a^{\frac{1-\gamma}{2}} .$$

We conclude this section by examining the normal crack displacement. The calculation of the displacement of the crack faces is tedious, though in principle it is easy to construct an expression for $u_y^-(x,0)$. In particular, from (6) it follows that

$$\hat{u}_y(p,0) = K_o^{-1}[H_i(p)]^{-1}|p|^{\gamma-1}\hat{\sigma}_{yy}(p,0)$$

and consequently

$$u_y(x,0) = K_o^{-1}[(H_i^{-1}(p)|p|^{\gamma-1})^{\vee} * \sigma_{yy}](x). \tag{34}$$

This calculation depends heavily on the explicit form of the time behavior of the shear modulus. Even for the power-law model, the Fourier transform in (34) can only be explicitly calculated for certain ranges of the parameters α and γ. A more efficient method of analyzing the crack profile and deducing the effect of the viscoelastic and spatial parameters is to examine the derivative of the normal displacement.

Consider first the power-law behavior in time, i.e., case (2). From (6) and (24) it follows that

$$\hat{u}_y(p,0) = C_o e^{-i\alpha\frac{\pi}{2}\mathrm{sgn}(p)} \left\{ \frac{\hat{\sigma}_{yy}^-}{|p|^{1+\alpha-\gamma}} - \frac{k(p)}{i\pi|p|^{\frac{1+2\alpha-\gamma}{2}}} \int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}} \frac{d\tau}{\tau-p} \right\} \tag{35}$$

where

$$C_o = \frac{q(1-\nu)\, I \sin(\frac{q\pi}{2})\, y_c^{\gamma}}{2\Gamma(\gamma+2)\cos(\frac{\gamma\pi}{2})\,\mu_c (vt_c)^{\alpha}\Gamma(1-\alpha)} \; .$$

Three cases need to be distinguished; case 2(a): $(1+2\alpha-\gamma)/2 > 1$, case 2(b): $(1+2\alpha-\gamma)/2 = 1$, and case 2(c): $(1+2\alpha-\gamma)/2 < 1$. Note that for 2(a) and 2(b) it follows that $\alpha > \gamma$ while for 2(c), $\alpha > \gamma$ and $\alpha < \gamma$ are both possibilities.

Multiply (35) by $-ip$ to obtain

$$\hat{u}_{y,x}(p) = U_1(p) + U_2(p) \tag{36}$$

where

$$U_1(p) = \frac{-iC_o e^{-i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)}}{|p|^{\alpha-\gamma}}\,\mathrm{sgn}(p)\,\hat{\sigma}_{yy}^{-}(p)$$

and

$$U_2(p) = \frac{C_o e^{-i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)}}{|p|^{\frac{2\alpha-\gamma-1}{2}}}\,k(p)\,\mathrm{sgn}(p)\,H[-\hat{\sigma}_{yy}^{-}\hat{\beta}](p)$$

with $\hat{\beta}$ defined in (28).

In case 2(a), $u_{y,x}$ may be obtained by Fourier inversion of (36). In fact for this case it is possible to write

$$U_1(p) = \hat{r}(p)\,\hat{\sigma}_{yy}^{-} \tag{37}$$

and

$$U_2(p) = \hat{s}(p)\,[i\,\mathrm{sgn}(\tau)(\sigma_{yy} * \beta)]^{\wedge}$$

where

$$r(x) = \frac{-C_o\Gamma(1-\alpha+\gamma)}{\pi} \frac{1}{|x|^{1-\alpha+\gamma}} \begin{cases} \cos(\frac{\gamma\pi}{2}) & x > 0 \\ \cos(\frac{\gamma\pi}{2} + \alpha\pi) & x < 0 \end{cases}$$

and

$$s(x) = \frac{-C_o \; (\frac{3-2\alpha+\gamma}{2})}{2\pi|x|^{\frac{3-2\alpha+\gamma}{2}}} \begin{cases} 0 & x > 0 \\ e^{-i\frac{\pi}{4}}[e^{i\pi(\frac{\gamma}{4}-\alpha)} + e^{-i\pi(\frac{3\gamma}{4}-\alpha)}] & x < 0 \end{cases} .$$

Hence one obtains the following expression for the slope to the crack displacement,

$$u^-_{y,x}(x,0) = \int_{-\infty}^{0} \sigma^-_{yy}(u)\, r(x-u)\, du + i\int_{-\infty}^{0}\sigma^-_{yy}(u)\int_{x}^{\infty} \mathrm{sgn}(\omega)\, s(x-\omega)\, \beta(\omega-u)\, d\omega du. \quad (38)$$

To discover the nature of the crack profile at its tip compute $\lim_{x\to 0} u^-_{y,x}$. This yields

$$u^-_{y,x}(0^-) = \frac{-C_o\Gamma(1-\alpha+\gamma)\cos(\frac{\gamma\pi}{2})}{\pi} \int_{-\infty}^{0} \sigma^-_{yy}(u) \frac{du}{|u|^{1-\alpha+\gamma}}$$

$$- \frac{C_o}{2\pi^2}\Gamma(\frac{3-2\alpha+\gamma}{2})\Gamma(\frac{1+\gamma}{2})[\cos\pi\alpha + \cos\pi(\gamma-\alpha)]\int_{-\infty}^{0}\sigma^-_{yy}(u)\int_{0}^{\infty}\frac{d\omega}{\omega^{\frac{3-2\alpha+\gamma}{2}}(\omega-u)^{\frac{1+\gamma}{2}}}\, du. \quad (39)$$

Since

$$\int_0^{\infty}\sigma^-_{yy}(u)\int_0^{\infty}\frac{d\omega}{\omega^{\frac{3-2\alpha+\gamma}{2}}(\omega-u)^{\frac{1+\gamma}{2}}} = \frac{\Gamma(1+\gamma-\alpha)\,\Gamma(\frac{2\alpha-1-\gamma}{2})}{\Gamma(\frac{1+\gamma}{2})}\int_{-\infty}^{0}\sigma^-_{yy}(u)\frac{du}{|u|^{1-\alpha+\gamma}},$$

(39) may be simplified to show $u^-_{y,x}(0^-) = 0$. Thus, in case 2(a), the crack surfaces close smoothly to form a cusp, in spite of the presence

of a singular normal stress.

In case 2(b) the crack tip is no longer cusp-like but the tangent to the crack tip has finite slope. It is apparent that for $(1+2\alpha-\gamma)/2 = 1$ (37) is still valid while

$$U_2(p) = C_o e^{-i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)} k(p)\,\mathrm{sgn}(p)\,H[-\hat{\sigma}^-_{yy}\hat{\beta}](p).$$

In this case Fourier inversion of (36) yields

$$u^-_y(x,0) = \int_{-\infty}^{0} \sigma^-_{yy}(u)\,r(x-u)\,du - \tag{40}$$

$$- \frac{C_o}{2\pi^2}\,\Gamma(\alpha)\sin(\alpha\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{0}^{\infty} \frac{e^{-ip(x+u+\omega)}\,\mathrm{sgn}(u+\omega)\,\sigma^-_{yy}(u)}{\omega^{\alpha}}\,d\omega\,du\,dp.$$

From (40) it easily follows that the tangent to the crack tip has finite slope, corresponding to the crack faces meeting at a sharp corner.

The case of a homogeneous viscoelastic medium with a power-law time dependent shear modulus corresponds to $\gamma = 0$ in cases 2(a) and 2(b). In case 2(a), $\alpha > 1/2$ and the crack profile is cusp-like, while for 2(b), $\alpha = 1/2$ and the crack surfaces meet at a sharp corner. These results agree with the findings of Walton and Nachman (1979).

In case 2(c) the tangent to the crack displacement becomes vertical at the crack tip. Formulas analogous to (38) and (39) can be derived by Fourier inversion of (36). However, the calculations are lengthy. If an asymptotic analysis of $u^-_{y,x}$ is carried out, the nature of the crack profile is more easily found.

If $\alpha > \gamma$, $U_1 \in L^1(\mathbb{R})$, otherwise

$$U_1(p) \sim \frac{C_o f(0)}{|p|^{1+\alpha-\gamma}} e^{-i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)} \quad \text{as } |p| \to \infty. \tag{41}$$

To determine the asymptotic behavior of U_2 at $\pm\infty$, first use the identity $1/(\tau-p) = -1/p+\tau/(p(\tau-p))$ to write

$$H[-\hat{\sigma}^-_{yy}\hat{\beta}] = \frac{-1}{\pi p}\int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}}\,d\tau + G(p)$$

where $G(p) = O(|p|^{-(\frac{3+\gamma}{2})})$ as $|p| \to \infty$. From this it follows that

$$U_2(p) \sim -\frac{C_o e^{-i\alpha\frac{\pi}{2}\,\mathrm{sgn}(p)}\,k(p)}{\pi|p|^{\frac{1+2\alpha-\gamma}{2}}} \int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}}\,d\tau$$

and hence

$$U_2(p) \sim -\frac{C_o}{\pi}\int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}}\,d\tau \; \frac{1}{|p|^{\frac{1+2\alpha-\gamma}{2}}} \begin{cases} e^{-i\alpha\frac{\pi}{2}} & p \to \infty \\ e^{i\frac{\pi}{2}(1+\alpha-\gamma)} & p \to -\infty \end{cases} \tag{42}.$$

From (41), (42), and the Abelian theorems for the Fourier transform, it follows that the dominant term in the asymptotic expansion of $u^-_{y,x}$ for x near 0 is given by

$$u^-_{y,x} \sim -\frac{C_o}{4\pi^2}\int_{-\infty}^{\infty} \frac{h(\tau)}{k(\tau)|\tau|^{\frac{1-\gamma}{2}}}\,d\tau \left\{ \int_0^{\infty} \frac{e^{-i\alpha\frac{\pi}{2}} e^{-ixp}}{p^{\frac{1+2\alpha-\gamma}{2}}}\,dp + \int_{-\infty}^{0} \frac{e^{i\frac{\pi}{2}(1+\alpha-\gamma)} e^{-ixp}}{|p|^{\frac{1+2\alpha-\gamma}{2}}}\,dp \right\}$$

$$= \frac{C_o}{\pi^2}\,\Gamma\left(\frac{1-2\alpha+\gamma}{2}\right)\Gamma\left(\frac{1+\gamma}{2}\right)[\cos\pi\alpha+\cos\pi(\alpha-\gamma)]\frac{1}{|x|^{\frac{1-2\alpha+\gamma}{2}}}\int_{-\infty}^{0}\frac{\sigma_{yy}^{-}(s)}{|s|^{\frac{1+\gamma}{2}}}\,ds. \tag{43}$$

The purely elastic result is recovered by setting $\alpha=0$ in (43). The homogeneous case is then found by letting $\gamma \to 0$. This yields that for x near 0,

$$u_{y,x}^{-} \sim -\frac{(1-\nu)}{\pi\mu_c}\frac{1}{|x|^{1/2}}\int_{-\infty}^{0}\frac{\sigma_{yy}^{-}(s)}{|s|^{1/2}}\,ds,$$

which agrees with the previously known result (Willis, 1967).

From (38), (40), and (43) it is apparent that the crack profile, unlike the stress field, depends on both the elastic and viscoelastic parameters γ and α. The larger the value of γ, and hence the softer the material near the crack plane, the greater the crack opening. The effect of the viscoelastic parameter α is to reduce the crack displacements. In particular, when the viscoelastic effects are dominant, i.e., cases 2(a) and 2(b), the blunt profile associated with a vertical tangent at the crack tip is transformed into a crack opening that is cusp-like or has finite slope.

One final remark is in order. The simple explicit expressions for $u_{y,x}^{-}$ given by (38), (40), and (43) cannot be obtained for a general time-dependent shear modulus. However, given the monotonicity and smoothness assumptions discussed previously, it is not difficult to adapt the methods employed for the pure power-law case in order to prove that a vertical tangent at the crack-tip is predicted by the more general models.

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Figure Caption for Figure 1: Image in ¢ of $H_1(z)$

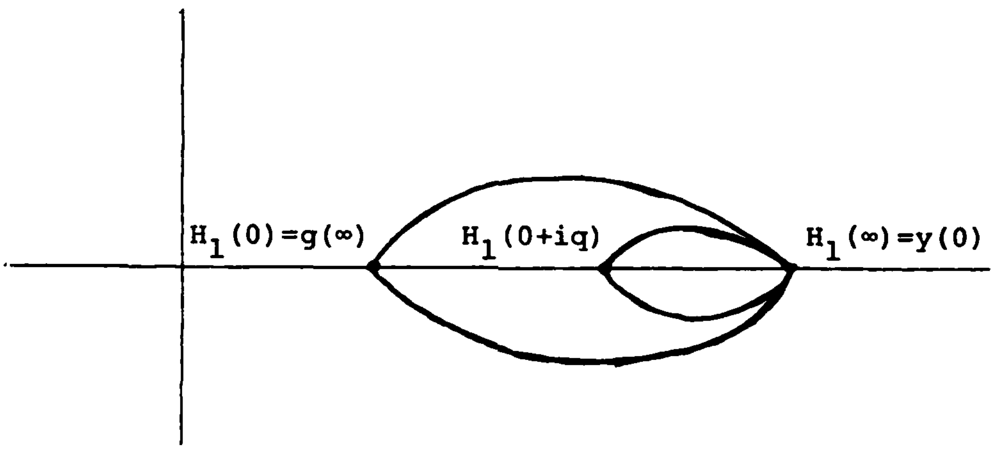

Fig. 1 Image in ¢ of $H_1(z)$

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